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FIXED POINT RESULTS FOR PARAMETRIC B-METRIC SPACE

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ABSTRACT

There are many generalization of metric space. Parametric metric space is the generalization of metric space too. Which was introduced and studied by Hussian (a new approach to metric space) in 2014. In present paper we prove two fixed point theorems based on injective mapping using contraction conditions.

Mathematics Subject Classification (MSC): *47H10, 54H25*

KEYWORDS: *Unique Fixed Point, Contraction, Parametric Metric Space, Injective Mapping*

INTRODUCTION

General Introduction

A metric on a nonempty set X is a mapping d: $X \times X[0. \infty]$satisfying the following properties:

- $d(x, y) = 0$ If and if $x = y$
- $d(x, y) = d(y, x)$
- $d(x, y) \leq d(y, z) + d(z, y)$; then the pair (X, d) is said to be a metric spaces.

The theory of metric spaces is the general theory which covers several branches of mathematical analysis, as real analysis, complex analysis, multidimensional calculus, etc. Due to which, existence and uniqueness of fixed points and common fixed points has become a subject of great concern. In the recent six decades many authors generalized the Banach contraction Principle by moderating the triangular inequality of a metric space as generalized metric space[see 2,5,7-8,14,22 and references therein], cone metric space[see 9 references therein], b metric space[see 2,3,4,6 references therein], cone b metric space[see9,10,11,14-22 references therein], rectangular metric space [see 17 references therein], cone rectangular metric space [see 12,17,18 references therein], are some of the generalizations of metric space introduced by different authors in past few decades. Analogue Banach contraction principle, Kannan contraction principle, Ciric contraction principle and lots of the existing fixed point theorems for various generalized contractions were proved in these generalized spaces.

Most of the generalization of metric space are Hausdorff topology but we can also find generalization of metric space which are not necessarily Hausdorff topology (see, ref. [13, 19, 22,]). Tarskian mathematician used non Hausdorff topology for programming language semantics used in computer science.

The purpose of this paper is to prove some fixed point theorems for contraction mapping inparametric metric spaces An example is also given to distinguish our results.

PRELIMINARIES

Proceeding to our main result, let we furnish some definition, proposition,properties & lemmas needed in sequel.

1. Let X be a non empty set and $d_p: X \times X \times (0, \infty) \to (0, \infty)$ be a map on X such that

$\forall\, x, y, z \in X$ and $t > 0$

- $d_p(x, y, t) = 0$ if and only if $x = y$
- $d_p(x, y, t) = d_p(y, x, t)$
- $d_p(x, y, t) \leq d_p(x, z, t) + d_p(z, y, t)$

Then d_p is called parametric metric and pair (X, d_p) is called parametric metric space.

- If $\lim_{n\to\infty}(x_n, x, t) = 0 \Rightarrow \lim_{n\to\infty} x_n = x$, for all $t > 0$ then sequence $\{x_n\}^{\infty}$ n=1 converses $x \in X$
- If $\lim_{n\to\infty}(x_n, x_m, t) = 0$ for all $t > 0$ then sequence $\{x_n\}^{\infty}$ n=1 is called Cauchy sequence.
- If every Cauchy sequence is convergent, then parametric metric space (X, d) is a completeparametric metric space.

Let (X, d) be a parametric metric space and $T: X \to X$ be a mapping, then We say T is acontinuous mapping at p in X, if for any sequence $\{x_n\}^{\infty}$ n=1 $\in X$ such that $\log_{n\to\infty} x_n =$

$x \Rightarrow \log_{n\to\infty} Tx_n = Tx$.

DEFINITION

Let X be a nonempty set, $s \geq 1$be a real number and $P_b: X \times X \times (0; 1) \in [0; 1]$; such that

- $P_b(x, y, t) = 0$ if and only if $x = y$
- $P_b(x, y, t) = P_b(y, x, t)$
- $P_b(x, y, t) \leq s[P_b(x, z, t) + P_b(z, y, t)\ ] \forall\, x, y, z \in X$ and $t > 0\ and\ s \geq 1$

Then, P_b is called parametric b-metric on X and (X, P_b) is called parametric b-metric spaces.

The following definitions and results will be needed in the sequel which can be

MAIN RESULT

The objective of this paper is to prove some new fixed point theorems in parametric metric space. This paper is divided in two sections. In Section I and II we prove theorems on parametric metric spaces.

SECTION I

Theorem 2

Let (X, d) be a complete parametric metric space and $T_p: X \to X$ be an injective mapping satisfying the condition

$$(2.1)d(T_p x, T_p y, t) \leq \alpha d(x, y, t) + \beta \left(\frac{d(x,T_p x,t)\,.d(y,T_p y,t)}{d\{x,y\}+d(x,T_p x,t)}\right) + \gamma \left(\frac{d(x,T_p x,t).d(x,T_p y,t)}{d\{x,y\}+d(x,T_p x,t)+2\,d(y,T_p y,t)}\right)$$

$\forall\, t \in [0,1)$; $\alpha, \beta, \gamma > 0$; $x, y \in X$ & $x \neq y$ have a fixed point if $\alpha + \beta\, 2 + \gamma\, 2 < 1$ and moreover a unique fixed point if $\alpha + \gamma < 1$.

Proof

Let ss$x0 \in X$, Define iterative sequence$\{xn\}n=1 \infty$ follows: $Tpxn = xn+1$ for n = 1, 2, 3, …. If for some n, $Tpxn = xn$, then xn is the fixed point. Otherwise$Tpxn \neq xn$, using inequality

$$d(x_{n+1}, x_{n+2}, t) = d(T_p x_n, T_p x_{n+1}, t)$$

$$\leq \alpha d(x_n, x_{n+1}, t) + \beta \left(\frac{d(x_n, T_p x_n, t).\, d(x_{n+1}, T_p x_{n+1}, t)}{d(x_n, x_{n+1}, t) + d(x_n, T_p x_n, t)}\right) + \gamma \left(\frac{d(x_n, T_p x_n, t).\, d(x_n, T_p x_{n+1}, t)}{d(x_n, x_{n+1}, t) + d(x_n, x_{n+1}, t) + 2d(x_{n+1}, x_{n+2}, t)}\right)$$

$$d(x_{n+1}, x_{n+2}, t) \leq \alpha d(x_n, x_{n+1}, t) + \beta \left(\frac{d(x_n, x_{n+1}, t).\, d(x_{n+1}, x_{n+2}, t)}{d(x_n, x_{n+1}, t) + d(x_n, x_{n+1}, t)}\right) + \gamma \left(\frac{d(x_n, x_{n+1}, t).\, d(x_n, x_{n+2}, t)}{d(x_n, x_{n+1}, t) + d(x_n, x_{n+1}, t) + 2\,.\, d(x_{n+1}, x_{n+2}, t)}\right)$$

$$d(x_{n+1}, x_{n+2}, t) \leq \alpha d(x_n, x_{n+1}, t) + \beta \left(\frac{d(x_n, x_{n+1}, t).\, d(x_{n+1}, x_{n+2}, t)}{2.\, d(x_n, x_{n+1}, t)}\right) + \gamma \left(\frac{d(x_n, x_{n+1}, t).\, d(x_n, x_{n+2}, t)}{2.\, d(x_n, x_{n+1}, t) + 2\, d(x_{n+1}, x_{n+2}, t)}\right)$$

$$(x_{n+1}, x_{n+2}, t) \leq \alpha d(x_n, x_{n+1}, t) + \beta \left(\frac{d(x_{n+1}, x_{n+2}, t)}{2.}\right) + \gamma \left(\frac{d(x_n, x_{n+1}, t).}{2.}\right)$$

$$\left(1 - \frac{\beta}{2}\right) d(x_{n+1}, x_{n+2}, t) \leq \left(\alpha + \frac{\gamma}{2}\right) d(x_n, x_{n+1}, t)$$

$$d(x_{n+1}, x_{n+2}, t) \leq \frac{\left(\alpha + \frac{\gamma}{2}\right)}{\left(1 - \frac{\beta}{2}\right)} d(x_n, x_{n+1}, t)s$$

$$d(x_{n+1}, x_{n+2}, t) \leq k\, d(x_n, x_{n+1}, t)$$

$\forall\, t \in [0,1) and\ k = \frac{\left(\alpha+\frac{\gamma}{2}\right)}{\left(1-\frac{\beta}{2}\right)} < 1 \Rightarrow \alpha + \frac{\beta}{2} + \frac{\gamma}{2} < 1$. Therefore by successive iteration

$$d(x_{n+1}, x_{n+2}, t) \leq k^n\, d(x_0, x_1, t)$$

As we know if $\{xn\}n\to\infty$ be a sequence in parametric space(X, d) such that $d(xn+1, xn+2, t) \leq k\ d(xn, xn+1, t)$ $\forall\ t \in [0,1)$ & n = 1,2,3, … then$\{xn\}n\to\infty$ is a Cauchy sequence in (X, d). Since (X, d) is a complete parametric space; $\{xn\}n\to\infty$ converses. Let $x* \in X$, then $limn\to\infty\ xn \to x*$. Again Tp is continuous, therefore

$$T_p x^* = T_p(lim_{n\to\infty}\, x_n) = \ lim_{n\to\infty}\, T_p x_n = x^* \Rightarrow T_p x^* = x^*$$

Implies Tp has a fixed point $Tpx* = x*$ in X.

Now we will show that $x*$ is unique. for that suppose $y*$ is another fixed point therefore $Tpy* = y*$. Therefore by inequality (2.1) we have

$$d\left(T_p x^*, T_p y^*, t\right) \leq \alpha\, d(x^*, y^*, t) + \beta\left(\frac{d\left(x^* T_p, x^*, t\right).d\left(y^*, T_p y^*, t\right)}{d(x^*, y^*, t) + d\left(x^*, T_p, x^*, t\right)}\right) + \gamma\left(\frac{d\left(x^*, T_p x^*, t\right).d\left(x^*, T_p y^*, t\right)}{d(x^*, y^*, t) + d\left(x^*, T_p, x^*, t\right) + 2d\left(y^*, T_p, y^*, t\right)}\right)$$

$$d(x^*, y^*, t) \leq \alpha\, d(x^*, y^*, t) + \beta\left(\frac{d(x^*, x^*, t).d(y^*, y^*, t)}{d(x^*, y^*, t) + d(x^*, x^*, t)}\right) + \gamma\left(\frac{d(x^*, x^*, t).d(x^*, y^*, t)}{d(x^*, y^*, t) + d(x^* x^*, t) + 2d(y^*, y^*, t)}\right)$$

$$d(x^*, y^*, t) \leq \alpha\, d(x^*, y^*, t)$$

$$d(x^*, y^*, t) \leq \alpha\, d(x^*, y^*, t)$$

$$\Rightarrow (1-\alpha) d(x^*, y^*, t) \leq 0$$

$\Rightarrow d(x^*, y^*, t) = 0$ Since $\alpha < 1 \Rightarrow x^* = y^*$. Hence T_p has a unique point.

SECTION II

Theorems

Let (X, Tp) be a complete parametric b- metric space and Tp: $X \to X$ be a orbitally continuous self-map satisfying the condition

$$(3.1)\left(T_p x, T_p y, t\right) \leq \alpha\, Max\{d(x, y, t), \frac{d(x,T_p x,t)d(y,T_p y,t)}{d(x,y,t)}, \frac{d(x,T_p y,t)d(y,T_p x,t)}{d(x,y,t)}, \frac{d(x,T_p x,t)d(x,T_p y,t)}{2d(x,y,t)}\}$$

$\forall\ t \in [0,1);\ \alpha > 0; x, y \in X\ \&\ x \neq y$ and $\alpha \in [0,1]$,then T_p has a unique fixed point.

Proof

Let $x0 \in X$, Define iterative sequence $\{xn\}n$=1 ∞ follows: $Tpxn = xn$+1 for n = 1, 2, 3, …. If for some n $Tpxn = xn$, then xn is the fixed point. Otherwise$Tpxn \neq xn$, using in equality(3.1)

$$d(x_{n+1}, x_{n+2}, t) = d(T_p x_n, T_p x_{n+1}, t)$$

$$\leq \alpha\, Max\{d(x_n, x_{n+1}, t), \frac{d(x_n, T_p\, x_n, t)d(x_{n+1}, T_p x_{n+1}, t)}{d(x_n, x_{n+1}, t)},$$

$$\frac{d(x_n, T_p x_{n+1}, t)d(x_{n+1}, T_p x_n, t)}{d(x_n, x_{n+1}, t)}, \frac{d(x_n, T_p x_n, t)d(x_n, T_p x_{n+1}, t)}{2d(x_n, x_{n+1}, t)}\}$$

$$\leq \alpha\, Max\{d(x_n, x_{n+1}, t), \frac{d(x_n, x_{n+1}, t)d(x_{n+1}, x_{n+2}, t)}{d(x_n, x_{n+1}, t)},$$

$$\frac{d(x_n, x_{n+2}, t)d(x_{n+1}, x_{n+1}, t)}{d(x_n, x_{n+1}, t)}, \frac{d(x_n, x_{n+1}, t)d(x_n, x_{n+2}, t)}{2d(x_n, x_{n+1}, t)}\}$$

$$\leq \alpha\, Max\{d(x_n, x_{n+1}, t), \frac{d(x_n, x_{n+1}, t)d(x_{n+1}, x_{n+2}, t)}{d(x_n, x_{n+1}, t)}, 0, \frac{d(x_n, x_{n+1}, t)\{d(x_n, x_{n+1}, t) + d(x_{n+1}, x_{n+2}, t)\}}{2d(x_n, x_{n+1}, t)}$$

$$d(x_{n+1}, x_{n+2}, t) \leq \alpha\, Max\{(d(x_n, x_{n+1}, t), d(x_{n+1}, x_{n+2}, t), 0, \frac{d(x_n, x_{n+1}, t) + d(x_{n+1}, x_{n+2}, t)}{2}\}$$

$$\Rightarrow d(x_{n+1}, x_{n+2}, t) \leq \alpha\, d(x_n, x_{n+1}, t)$$

Therefore by successive iteration

$$d(x_{n+1}, x_{n+2}, t) \leq \alpha^n\, d(x_0, x_1, t)$$

$$d(x_{n+1}, x_{n+2}, t) \leq \alpha^n\, d(x_0, x_1, t)$$

As we know if $\{xn\}n\to\infty$ be a sequence in parametric space(X, d) such that $d(xn+1, xn+2, t) \leq \alpha n\, d(x0, x1, t)$

$\forall\, t \in [0,1)$ & n = 1,2,3, … then$\{xn\}n\to\infty$ is a Cauchy sequence in (X, d). Since (X, d) is a complete parametric space; $\{xn\}n\to\infty$ converses. Let $x* \in X$, then $limn\to\infty\, xn \to x*$. Again Tp is continuous, therefore

$$T_p x^* = T_p(lim_{n\to\infty}\, x_n) = lim_{n\to\infty}\, T_p x_n = x^* \Rightarrow T_p x^* = x^*$$

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$$d(T_p x^*, T_p y^*, t)$$

$$\leq \alpha\, Max\{d(x^*, y^*, t), \frac{d(x^*, T_p x^*, t)d(y^*, T_p y^*, t)}{d(x^*, y^*, t)}, \frac{d(x^*, T_p y^*, t)d(y^*, T_p x^*, t)}{d(x^*, y^*, t)}, \frac{d(x^*, T_p x^*, t)d(x^*, T_p y^*, t}{2d(x^*, y^*, t)}$$

$$d(T_p x^*, T_p y^*, t)$$

$$\leq \alpha\, Max\{d(x^*, y^*, t), \frac{d(x^*, x^*, t)d(y^*, y^*, t)}{d(x^*, y^*, t)}, \frac{d(x^*, y^*, t)d(y^*, x^*, t)}{d(x^*, y^*, t)}, \frac{d(x^*, x^*, t)d(x^*, y^*, t)}{2d(x^*, y^*, t)}\}$$

$$d(T_p x^*, T_p y^*, t) \leq \alpha\, Max\{d(x^*, y^*, t), 0, d(x^*, y^*, t), 0\}$$

$$d(x^*, y^*, t) \leq \alpha d(x^*, y^*, t)$$

$$\Rightarrow (1 - \alpha)d(x^*, y^*, t) \leq 0$$

$\Rightarrow d(x^*, y^*, t) = 0$ Since $\alpha > 1 \Rightarrow x^* = y^*$. Hence T_p has a unique point.

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